Preface

The thesis is the outcome of my research work on geometry and topology. This work is specifically in the field of homotopy theory and theory of vector fields. This thesis consists of answers to the five research problems that are considered both in geometry and topology including the homeomorphism problem. The prerequisites are an elementary knowledge of algebra, analysis, general topology, algebraic topology and theory of manifolds.

The thesis provides a systematic and comprehensive development in the extension of the fundamental and higher homotopy group called Core fundamental groupoid and also a method to obtain left-invariant vector fields on some smooth manifolds which admits a regular Lie group action. More precisely, we have two main objectives: The first one in topology is to solve the homeomorphism problem by giving a sufficient new invariant by extending the theory of homotopy. The second one is to provide a general method to obtain left-invariant vector fields on some smooth manifolds which admits an action.

The first objective of obtaining a sufficient invariant is resolved by introducing Core fundamental groupoid which is a topological groupoid. This leads to study on covering maps and Core fundamental groupoid, Core fundamental groupoid bundle, section on Core fundamental groupoid bundle and method of obtaining the left-invariant sections of Core fundamental groupoid of some topological spaces which admits a regular topological group action. The second one is followed by the successive group actions on first on a manifold and second on its tangent bundle by a Lie group and its transformation group on manifold respectively. This thesis has eight chapters that contain explicitly as contains the following:

The first chapter contains the preliminaries to the entire thesis work. It contains fundamental notions and basic properties of groups, groupoid, topology, manifolds, group action, bundles and category.

The second chapter contains a new notion called Core fundamental groupoid. Section 2.2 introduces Core fundamental groupoid, it is an extension of the fundamental group (Poincare group). It is defined by the disjoint union of the fundamental groups at points of M, and denoted by $\bar{\pi}_1 M = \bigcup_{x \in M} \pi_1(M, x)$. Here groupoid is used in algebraic sense, not as a category (Groupoid [81] is a category in which every morphism is invertible). It is proved that, for a given topological space M, then $\bar{\pi}_1 M$ forms a groupoid. Induced homomorphism is extended to induced groupoid homomorphism on respective Core fundamental groupoids for a continuous map and thus its similar kind of properties like induced homomorphism placed in this section. It also contains induced base map for a groupoid homomorphism $T : \pi_1 M \to \pi_1 N$ and is defined by a map $\mathfrak{b}_T : M \to N$ such that $p_N \circ T = \mathfrak{b}_T \circ p_M$, and some of basic consequences. It contains a main result that the fundamental groupoid is a topological invariant but not sufficient. In the section 2.3 the fundamental groupoid is shown as a topological groupoid which is a sufficient topological invariant and thereby provides a characterization to the homeomorphism. That is stated in a proposition as "Let M, N be two topological spaces and $f : M \to N$ be a map then f is a homeomorphism if and only if $f_\# : (\bar{\pi}_1 M, \mathfrak{I}_{p_M}) \to (\bar{\pi}_1 N, \mathfrak{I}_{p_N})$ is a topological groupoid isomorphism". Section 2.4 contains some categorical view points regarding Core fundamental groupoid and a new notion of 'same Core fundamental groupoid type relation' and also its application to a characterization to the homeomorphism.

The third chapter contains another new notion called Core higher homotopy groupoid. Section 3.2 introduces Core higher homotopy groupoid, which is an extension of the fundamental groupoid and higher homotopy group. It is defined by the disjoint union of the higher homotopy groups at points of M, and denoted it by $\bar{\pi}_k M = \bigcup_{x \in M} \pi_k(M, x)$. It is proved that, for a given M a topological space then $\bar{\pi}_k M$ forms a groupoid for each $k \in \mathbb{N}$. Induced homomorphism is extended to induced groupoid homomorphism on respective Core fundamental groupoids for a continuous map and thus its similar kind of properties like induced homomorphism are placed in this section. It also contains induced base map for a groupoid homomorphism $T : \pi_k M \to \pi_k N$ on k-th homotopy groupoids and is defined by a map $\mathfrak{b}_T : M \to N$ such that $p_N \circ T = \mathfrak{b}_T \circ p_M$, and some of the basic consequences. It contains a main result that the higher homotopy groupoid is a topological invariant but not sufficient. In the section 3.3 the higher homotoy groupoid is shown as a topological groupoid which is a sufficient topological invariant and thereby provides a characterization to the homeomorphism. That is stated in a proposition as "Let M, N be two topological spaces, $f : M \to N$ be a map and k be a natural number then f is a homeomorphism if and only if $f_\# : (\bar{\pi}_k M, \mathfrak{I}_{p_M}) \to (\bar{\pi}_k N, \mathfrak{I}_{p_N})$ is a topological groupoid isomorphism". Section 3.4 contains some categorical view points regarding Core higher homotopy groupoid and a new notion of same 'Core homotopy groupoid type relation' also its application to a characterization to the homeomorphism.

The fourth chapter contains study on covering projection and Core fundamental groupoid. Section 4.2 provides some characterization of the simply connectedness with the concept of Core fundamental groupoid. It also contain the basic consequences on induced groupoid homomorphisms of covering projections. Section 4.2 provides a method of obtaining covering spaces to a connected, locally path-connected, and semi locally simply connected space and for each wide subgroupoids of it.

The fifth chapter contains a theory of Core fundamental groupoid bundle. It emphasizes bundle structure, section and relatedness on sections of a Core fundamental groupoid bundle of a topological space. It provides explicit description of the bundle, fibre bundle, principal G-bundle structure on the Core fundamental groupoid in respective cases. Section 5.3 contains sections of Core fundamental groupoid bundle and pushforword and pullback maps on section space of respective topological spaces of a homeomorphism. Section 5.4 contains Relatedness on the both homeomorphisms on space and sections of its Core fundamental groupoid bundle i.e. section related homeomorphism and homeomorphism section. Rel operation is extensively discussed in this section and an interrelationship is built between some subgroups of homeomorphisms on a space and some subgroups of sections of the Core fundamental groupoid bundle.

The sixth chapter contains a method of obtaining left-invariant sections of the Core fundamental groupoid bundle for some topological spaces which admits a regular topological group action.

The seventh chapter contains a method which yields vector fields on some smooth manifolds. A method is introduced to obtain left-invariant smooth vector fields on some smooth manifolds which admits a regular Lie action. This sections also contain results on support and smooth frame on such manifolds. Section 7.3 contains Relatedness on both diffeomorphisms and vector fields on the smooth manifolds i.e. vector field related diffeomorphisms and diffeomorphism vector fields. Rel operation is extensively discussed in this section and an interrelationship is built between some subgroups of homeomorphisms on a manifold and some subspaces of set of all vector fields on it.

The eight chapter contains some of the important applications that are coming from the research work carried out in the previous chapters. This section highlights the classification problem that is classification of spaces up to homeomorphism by emphasizing a characterization to homeomorphism by Core fundamental groupoid. Also it highlights some important results placed in the previous chapters. Finally, section 8.3 provides overall outcomes as a conclusion that explains and concludes our research work.

It is to be note that, we have used the notation μ or μ_i for $i = 1, 2$ for group action in all most all the chapters. Therefore, consider them and their consequences in the respective chapters independently.